Boundedness of the Range of a Strategy-Proof Social Choice Function

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Abstract. For the provision of $m \geq 1$ divisible public goods, relatively weak restrictions on the domain of a strategy-proof social choice function are identified that ensure that its range is bounded. Domain restrictions are also identified for which strategy-proofness implies that the range and the option sets of a social choice function are compact. To illustrate the usefulness of these results, it is shown how a theorem about generalized median voter schemes due to Barberà, Massó, and Serizawa can be established without their assumption that the range of a social choice function is compact provided that the tops of the preferences are not restricted to be finite.

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1. Introduction

We consider the strategy-proof provision of $m \geq 1$ divisible public goods. The set of alternatives over which preferences are defined is $X^* = (\mathbb{R}_+ \cup \{\infty\})^m$. We assume that everybody has the same set of admissible preferences, what Barberà (2011) calls a *common domain*. This assumption is natural if the alternatives have no private components, as is the case here.

When the domain is common, it is sometimes assumed that the set of alternatives or the range of a social choice function is compact. Prominent examples include Barberà, Gul, and Stacchetti (1993) and Barberà, Massó, and Serizawa (1998), both of which consider alternatives that lie in a multi-dimensional Euclidean space. In the former, the set of alternatives is a finite grid and in the latter, the range of the social choice function is assumed to be a full-dimensioned compact set. For a number of domains of continuous preferences, strategy-proofness implies that the range is closed. In other words, closure of the range does not need to be assumed *a priori*. See, for example, Barberà and Peleg (1990), Barberà and Jackson (1994), Le Breton and Weymark (1999), and Zhou (1991).

For an unbounded set of alternatives, these results leave open the question of when strategy-proofness implies that the range of the social choice function is bounded. Here, we show that if the set of alternatives is X^* and the domain of the social choice function includes all of the continuous, additively separable, single-peaked preference orderings with a unique most-preferred alternative on X^* at ∞^m, then strategy-proofness implies that the range is bounded.[1] We also show that the range is bounded if for each public good, the domain includes the preference that ranks alternatives solely on how much of that good is provided with more preferred to less. In the unidimensional case, in order to satisfy either of these domain conditions, only a single preference must be in the domain in order for the range to be bounded—the preference that is monotonically increasing. Thus, boundedness of the range of a strategy-proof social function holds for a wide range of domains of interest.

By combining our theorem with what is already known about the closure of the range of a strategy-proof social choice function, we are also able to identify restrictions on the domain that ensure that the range is compact.

[1] The unique most-preferred alternative of a preference (if it exists) is called the preference's *top*.

An example of a domain for which this is the case is the domain of continuous, additively-separable, single-peaked preferences on X^* provided that the preference tops are permitted to be infinite in any dimension.

In order to illustrate the usefulness of our results, it is shown how a theorem about generalized median voter schemes due to Barberà, Massó, and Serizawa (1998) for multidimensional, continuous, single-peaked preferences can be established without their assumption that the range of a social choice function is compact provided that the tops of preferences are not restricted to be finite.

It is important for the proof of our bounded range results that we allow the top of a preference to be infinite in some or all dimensions. However, the chosen alternative must not have any infinite components. Both of these assumptions are quite natural. The former permits a preference to be monotonically increasing in some or all dimensions. The latter requires the provision of any public good to be finite. By allowing for monotonic preferences, we depart from the standard assumption that the tops of preference are finite when the domain consists of profiles of unidimensional or multidimensional single-peaked preferences.[2]

Traditional proofs of the Gibbard (1973)–Satterthwaite (1975) Theorem for strategy-proof social choice functions employ steps in which a new profile of preferences is constructed by moving the ranking of an alternative in some individuals' preferences so that there is only one alternative preferred to it. For example, this is the case with the well-known proof of Reny (2001). This construction is not possible if the admissible preferences do not have a most-preferred alternative or if the set of alternatives is connected and the preferences are assumed to be continuous. Here, the set of alternatives is the connected set X^* and the preferences in our theorems are continuous.

A different way of determining the implications of strategy-proofness by itself or in combination with other desirable properties of a social choice function is to identify the structure of the social choice function's option sets. The *option set* generated by fixed preferences of a subgroup of individuals is the set of alternatives that can be chosen by the social choice function for some reports of the preferences of the other individuals. If there are n individuals of which the preferences of k of them are fixed, then the option set is the

[2]In the case of unidimensional single-peaked preferences, the generalized median social choice functions introduced by Moulin (1980) choose the median of the peaks (i.e., the tops) of the actual individuals and the fixed peaks of some "phantom" individuals. Moulin allows the phantom peaks to be infinite but not those of the real individuals.

range of an $(n-k)$-person social choice function defined on the preference profiles of those individuals whose preferences have not been predetermined. If $k=0$, the only option set is the range of the social choice function. For example, if there is a dictator, then the option set generated by the dictator's preference is the set of his or her most-preferred alternatives on the range of the social choice function and the option set generated by any subgroup that does not include the dictator is this function's range. The option set methodology has has been widely employed since Barberà and Peleg (1990) proved a version of the Gibbard–Satterthwaite Theorem for continuous preferences using it.[3]

We use the option set methodology to establish our results about the boundedness of the range of a strategy-proof social choice function. Because any option set is necessarily bounded if the range is bounded, our results also establish that all of the option sets are bounded. By combining this finding with what is already known about domain restrictions that imply that the option sets are closed, we are also able to identify restrictions on the domain that ensure that the option sets are compact.

2. The Model

The set of individuals is $N=\{1,\dots,n\}$, where $n\geq 2$. The set of public goods is $M=\{1,\dots,m\}$, where $m\geq 1$. Let $\mathbb{R}^*_+=\mathbb{R}_+\cup\{\infty\}$. The set of alternatives is $X^*=(\mathbb{R}^*_+)^m$. Thus, each good is perfectly divisible and can be provided in any nonnegative amount. We also consider the subset of alternatives $X=(\mathbb{R}_+)^m$ in which the alternatives are finite-valued.

A (weak) *preference relation* R is a binary relation on X^*, with xRy interpreted as meaning that x is weakly preferred to y. The corresponding strict preference and indifference relations are denoted by P and I, respectively. A preference relation R is an *ordering* if it is reflexive, complete, and transitive and it is *continuous* if $\{y\in X^*|xRy\}$ and $\{y\in X^*|yRx\}$ are both closed. It is *additively separable* if there exist functions $U_l\colon\mathbb{R}^*_+\to\mathbb{R}$, $l\in M$, such that for all $x,y\in X^*$, $xRy\leftrightarrow\sum_{l=1}^m U_l(x_l)\geq\sum_{l=1}^m U_l(y_l)$. For an additively separable preference, there is a well-defined marginal preference on each dimension.

[3] Option sets had been independently introduced by Barberà (1983) and Laffond (1980) but their usefulness was not generally appreciated until the publication of the Barberà–Peleg article.

For any $A \subseteq X^*$, the *top set on A* of a preference R is

$$\tau(R, A) = \{x \in A | xRy \text{ for all } y \in A\}.$$

If $\tau(R, A) = \{x\}$ for some $x \in A$, then x is called the *top* of R on A.

A preference ordering R on X^* is *single-peaked* if

(i) the top $\tau(R, X^*)$ of R on X^* is a singleton;

(ii) alternatives are less preferred the further they are from the top in any direction, where distance is measured by the L_1 (city block) norm.[4]

In one dimension, this is equivalent to the standard definition of single-peakedness if the top of a preference is permitted to be infinite. In higher dimensions, such a preference is usually referred to as being multidimensional single-peaked.

Public good $l \in M$ is *essential* for the preference R if there exist $x, y \in X^*$ with $x_l \neq y_l$ such that xPy. In other words, the preference must be sensitive to the quantity provided of this good. Each good is essential if the preference is single-peaked.

A preference R is *monotone* if for all $x, y \in X^*$, $x \gg y \to xPy$. That is, if x is larger than y in every component, then x is strictly preferred to y. Such a preference has a top on X^* at ∞^m and each public good is essential. For each $l \in M$, let $R_l^{\uparrow}$ denote the preference for which for all $x, y \in X^*$, $xRy \leftrightarrow x_l \geq y_l$. With $R_l^{\uparrow}$, only the lth public good is essential. When $m = 1$, we write $R^{\uparrow}$ instead of $R_1^{\uparrow}$. Note that $R^{\uparrow}$ is a continuous monotone preference.

A *profile* is an n-tuple of individual preference orderings $\mathbf{R} = (R_1, \ldots, R_n)$. Each individual is assumed to have the same set of admissible preferences $\mathcal{D}$. Thus, the set of admissible profiles is $\mathcal{D}^n$. The assumption that everybody has the same set of admissible preferences plays a fundamental role in our analysis.

A *social choice function* is a function $F \colon \mathcal{D}^n \to X$. Note that for each profile of preferences in the domain of F, a finite quantity of each public good is chosen. The *range* of F is

$$A^F = \{x \in X | F(\mathbf{R}) = x \text{ for some } \mathbf{R} \in \mathcal{D}^n\}.$$

[4]For a more formal definition, see Barberà, Massó, and Serizawa (1998).

For any set of individuals $G \subseteq N$ with $G \neq N$ and any profile $\mathbf{R} \in \mathcal{D}^n$, $\mathbf{R}_G$ denotes the *subprofile* of preferences of the individuals in G. We sometimes write $\mathbf{R}$ as $(\mathbf{R}_G, \mathbf{R}_{N\backslash G})$. If $G = \{i\}$ for some $i \in N$, we write R_i instead of $R_{\{i\}}$ and $\mathbf{R}_{-i}$ instead of $\mathbf{R}_{N\backslash\{i\}}$.

The social choice function is *strategy-proof* if there does not exist a profile $\mathbf{R} \in \mathcal{D}^n$ and an individual $i \in N$ such that $f(\bar{R}_i, \mathbf{R}_{-i}) P_i f(\mathbf{R})$ for some $\bar{R}_i \in \mathcal{D}$.

For the social choice function F, the *option set* generated by the subprofile $\mathbf{R}_G$ is

$$O^F_{N\backslash G}(\mathbf{R_G}) = \left\{x \in X | x = F(\mathbf{R}_G, \mathbf{R}_{N\backslash G}) \text{ for some } \mathbf{R}_{N\backslash G} \in \mathcal{D}^{n-|G|}\right\}.$$

This option set is the set of alternatives that are achievable given that the individuals in G have reported the preferences in $\mathbf{R}_G$.

For any $\mathbf{R}_G \in \mathcal{D}^{|G|}$, the *reduced social choice function* $F^{\mathbf{R}_G} \colon \mathcal{D}^{n-|G|} \to X$ is the $(n - |G|)$-person social choice function defined by setting

$$F^{\mathbf{R}_G}(\mathbf{R}_{N\backslash G}) = F(\mathbf{R}_G, \mathbf{R}_{N\backslash G}) \text{ for all } \mathbf{R}_{N\backslash G} \in \mathcal{D}^{n-|G|}.$$

The range of this function is the option set $O^F_{N\backslash G}(\mathbf{R_G})$. If $G = \varnothing$, then the option set is the range A^F. If a social choice function is strategy-proof, then so is any reduced social choice function obtained by fixing the preferences of a subgroup of the individuals.

3. Boundedness and Closure of the Range and the Option Sets

Le Breton and Weymark (1999, Proposition 1) have shown that any preference in the domain of a strategy-proof social choice function must have one or more alternatives that maximize it on the range. This result is used in the proof of our theorems about the boundedness of the range.

Lemma 1. *If the social choice function $F \colon \mathcal{D}^n \to X$ is strategy-proof, then for any any $R \in \mathcal{D}$, $\tau(R, A^F) \neq \varnothing$.*

Le Breton and Weymark established this lemma for any set of alternatives, not just the set X^* used here. It demonstrates that strategy-proofness by itself places a strong restriction on what the range can be, at least when there is a common domain. For example, if the set of alternatives is $\mathbb{R}^*_+$ and the monotonically increasing preference $R^\uparrow$ is in the domain, then the range must have a least upper bound.

The following lemma is a special case of Proposition 2 in Le Breton and Weymark (1999). It shows that if everybody agrees that some alternative is uniquely best on the range of a strategy-proof social choice function, then it must be chosen.

Lemma 2. *If the social choice function $F\colon \mathcal{D}^n \to X$ is strategy-proof, then for any any $\mathbf{R} \in \mathcal{D}^n$ for which there exists an $x \in A^F$ such that $\tau(R_i, A^F) = \{x\}$ for all $i \in N$, then $F(\mathbf{R}) = x$.*

Theorem 1 shows that the range of a strategy-proof social choice function must be bounded if the domain includes a sufficiently rich set of preferences for which every public good is essential.[5] This domain may, but need not, include some preferences for which not all goods are essential.

Theorem 1. *If $\mathcal{D}$ includes all of the continuous, additively separable, single-peaked preference orderings with a unique top on X^* at ∞^m and $F\colon \mathcal{D}^n \to X$ is a strategy-proof social choice function, then the range A^F of F is bounded.*

Proof. By way of contradiction, suppose that A^F is not bounded. Then there exists an $l \in M$ such that for each $\bar{x}_l \in \mathbb{R}_+$, there exists an $\hat{x} \in A^F$ such that $\hat{x}_l > \bar{x}_l$.

Given an arbitrary $\varepsilon > 0$, let $R^\varepsilon \in \mathcal{D}$ be such that for any $x, x' \in X^*$ for which $x_l > x'_l + \varepsilon$, we have $xP^\varepsilon x'$. The existence of such a preference can be confirmed by noting that the preference induced by a continuous, additively separable utility function of the form $\sum_{j=1}^m \lambda_j U_j(x_j)$ with each $\lambda_j > 0$ satisfies this property if $U_j(x_j)$ is increasing in x_j for all $j \in M$ and each of the λ_j for $j \neq l$ are chosen to be sufficiently close to 0.

Consider any $\bar{x} \in A^F$. Because A^F is unbounded in the lth dimension, there exists an $\hat{x} \in A^F$ such that $\hat{x}_l > \bar{x}_l$ and, hence, for which $\hat{x}P^\varepsilon\bar{x}$. Thus, $\tau(R^\varepsilon, A^F) = \varnothing$, which contradicts Lemma 1. □

If a preference ordering is continuous, additively separable, and single-peaked with a unique top at ∞^m, it is monotonically increasing in each dimension. Such a preference is excluded if, as is typically assumed, the domain only contains unidimensional or multidimensional single-peaked preferences whose tops are in X.

The assumption in Theorem 1 about what preferences must be in the domain is not very demanding. Consequently, it applies to many domains

[5] A subset of X^* is *bounded* if it is contained in a ball with finite radius.

that are of interest. For example, it applies to the domain of all single-peaked preferences on X^*. This assumption also applies to the domain of preferences considered in the proof of Theorem 1, that is, to the set of preferences induced by a continuous, additively separable utility function of the form $\sum_{j=1}^{m} \lambda_j U_j(x_j)$, where for all $j \in M$, $\lambda_j > 0$ and $U_l(x_l)$ is increasing in x_j. Such preferences are single-peaked. This set of preferences includes all the monotonic preferences with linear indifference contours, weighted Cobb–Douglas preferences, and preferences that put almost all weight on a single good, among others.

The m preferences $R_l^{\uparrow}$, $l \in M$, that regard just one public good as being essential are not single-peaked and, therefore, may not be in a domain that satisfies the domain assumption in Theorem 1. The boundedness of the range can also be established by only supposing that these m preferences are in the domain.

Theorem 2. *If $\mathcal{D}$ includes $R_l^{\uparrow}$ for all $l \in M$ and $F \colon \mathcal{D}^n \to X$ is a strategy-proof social choice function, then the range A^F of F is bounded.*

Proof. The proof of this result is almost identical to the proof of Theorem 1. Because the preference $R_l^{\uparrow}$ is monotonically increasing in the quantity of public good l, it can be used instead of the preference R^{ε} in the proof of that theorem. □

In the unidimensional case, additive separability is vacuous. Consequently, in this case, Theorem 1 implies that any strategy-proof social choice function whose domain contains the monotonically increasing preference $R^{\uparrow}$ must have a bounded range. The same conclusion also follows immediately from Theorem 2.

Corollary 1. *For $m = 1$, if $\mathcal{D}$ includes the preference $R^{\uparrow}$ and $F \colon \mathcal{D}^n \to X$ is a strategy-proof social choice function, then the range A^F of F is bounded.*

As has already been noted, for some domains, the range of a strategy-proof social choice function is closed. Because an option set is the range of a reduced social choice function, it then follows that all of the option sets are closed for such domains. The most general version of these results that has been established is obtained by combining Propositions 5 and 6 in

Le Breton and Weymark (1999).[6] The following lemma is an implication of their propositions.

Lemma 3. *If $\mathcal{D}$ includes a continuous preference ordering with a unique top at x for all $x \in X^*$ and $F\colon \mathcal{D}^n \to X$ is a strategy-proof social choice function, then for any $G \subseteq N$ with $G \neq N$ and any $\mathbf{R}_{N\setminus G} \in \mathcal{D}^{n-|G|}$, the option set $O_G^F(\mathbf{R}_{N\setminus G})$ is closed.*

In particular, by setting $G = \varnothing$, Lemma 3 shows that the range is closed if the assumptions of the lemma are satisfied.

Because all of the option sets are bounded if the range is bounded, by combining Theorem 1 with Lemma 3 it follows that all of the option sets of a strategy-proof social choice function are compact if the domain assumptions used in both of these results are satisfied.

Corollary 2. *If (i) $\mathcal{D}$ includes all of the continuous, additively separable, single-peaked preference orderings with a unique top on X^* at ∞^m, (ii) for all $x \in X^*$, $\mathcal{D}$ includes a continuous preference ordering with a unique top at x, and (iii) $F\colon \mathcal{D}^n \to X$ is a strategy-proof social choice function, then for any $G \subseteq N$ with $G \neq N$ and any $\mathbf{R}_{N\setminus G} \in \mathcal{D}^{n-|G|}$, the option set $O_G^F(\mathbf{R}_{N\setminus G})$ is compact.*

The conclusion of Corollary 2 also holds if (i) is replaced with the assumption that $R_l^{\uparrow} \in \mathcal{D}$ for all $l \in M$.

4. Generalized Median Voter Schemes

Barberà, Massó, and Serizawa (1998) have investigated the implications of strategy-proofness for a social choice function whose domain is the set of all profiles of multidimensional, continuous, single-peaked preferences whose tops are finite when its range is assumed to be compact and full-dimensioned. They have shown that with these assumptions, restricted to the subdomain in which the tops of the preferences are in the range, a strategy-proof social choice function must be a generalized median voter scheme that satisfies the intersection property.[7] Using Theorem 1, it is possible to obtain a version of

[6] They only require that the set of alternatives be a first-countable topological space and that for all x in the closure of the range there exists a continuous preference ordering in the domain with a unique top at x.

[7] Definitions of a generalized median voter scheme and the intersection property are provided below.

this theorem without assuming that the tops of preferences are finite or that the range is compact. Some further notation is needed in order to provide a formal statement of this result.

For all $l \in M$, a *right-coalition system* on a compact set $A_l \subseteq \mathbb{R}_+$ is a function $\mathcal{W}_l \colon A_l \to 2^N$ satisfying the following conditions:

(i) *Coalition Monotonicity:* For any $x_l \in A_l$ and any $W, W' \in 2^N$ for which $W \subseteq W'$, if $W \in \mathcal{W}_l(x_l)$, then $W \in \mathcal{W}_l'(x_l)$.

(ii) *Outcome Monotonicity:* For any $l \in M$, any $x_l, x_l' \in A_l$, and any $W \in \mathcal{W}_l(x_l)$, if $x_l' \leq x_l$, then $W \in \mathcal{W}_l(x_l')$.

(iii) *Voter Sovereignity:* For any $l \in M$ and any $x_l \in A_l$, $\mathcal{W}_l(x_l) \neq \varnothing$ and $\varnothing \notin \mathcal{W}_l(x_l)$. Furthermore, $\mathcal{W}_l(\underline{x}_l) = 2^N \setminus \varnothing$, where $\underline{x}_l = \min\{x_l | x_l \in A_l\}$.

(iv) *Upper Semi-continuity:* For any $l \in M$, any $W \subseteq N$, any $x_l \in A_l$, and any sequence $\{x_l^k\}_{k=1}^{\infty} \subseteq A_l$ for which $\lim x_l^k = x_l$, if $W \in \mathcal{W}_l(x_l^k)$ for all k, then $W \in \mathcal{W}_l(x_l)$.

A *family* $\mathcal{W}$ *of right-coalition systems* on a compact set $A \in \mathbb{R}_+^m$ is a collection $\{\mathcal{W}_l\}_{l \in M}$, where $\mathcal{W}_l$ is a right-coalition system on $\text{proj}_l(A)$ for all $l \in M$.

Let $\mathcal{D}^{CS}$ denote the set of all preferences that are continuous and single-peaked. For a social choice function $F \colon (\mathcal{D}^{CS})^n \to X$, let

$$\mathcal{D}^{CS^*} = \{R \in \mathcal{D}^{CS} | \tau(R, X^*) \in A^F\}$$

be the subset of $\mathcal{D}^{CS}$ for which the tops of the preferences are in the range of F.

A social choice function $F \colon (\mathcal{D}^{CS})^n \to X$ is a *generalized median voter scheme on the subdomain* $(\mathcal{D}^{CS^*})^n$ if its range A^F is compact and there exists a family $\mathcal{W}$ of right-coalition systems on A^F such that for all $\mathbf{R} \in (\mathcal{D}^{CS^*})^n$ and all $l \in M$,

$$F_l(\mathbf{R}) = \max\left\{x_l \in \text{proj}_l(A^F) | \left\{i \in N | \tau_l(R_i, A^F) \geq x_l\right\} \in \mathcal{W}_l(x_l)\right\}.$$

The set $\mathcal{W}_l(x_l)$ is the set of *winning coalitions* for dimension l at x_l. The quantity $F_l(\mathbf{R})$ of the lth public good is determined by starting at the upper endpoint of $\text{proj}_l(A^F)$ and decreasing the quantity of this good until the first quantity for which a winning coalition prefers at least this much of the good is reached.

For each profile of preferences in its domain, the alternative chosen by a generalized median voter scheme is determined component by component. If $m \geq 2$ and the range is not a product set, there is no guarantee that these choices result in a feasible alternative. To ensure that they do, the family of right-coalition systems must satisfy the intersection property. In order to define this property, we first need to introduce some further notation and definitions.

For any nonempty set $A \subseteq X^*$ and any $l \in M$, $\text{proj}_l(A)$ is the projection of A on the lth coordinate and

$$B_l(A) = \begin{cases} [\inf \text{proj}_l(A), \sup \text{proj}_l(A)] & \text{if } \sup \text{proj}_l(A) \text{ exists;} \\ [\inf \text{proj}_l(A), \infty) & \text{otherwise.} \end{cases}.$$

Let

$$B(A) = \prod_{l \in M} B_l(A).$$

By construction, $B(A)$ is closed.

For any set $A \in \mathbb{R}^m_+$ and any $x, y \in B(A)$, let $M^+(x, y) = \{l \in M | y_l > x_l\}$ and $M^-(x, y) = \{l \in M | y_l < x_l\}$. These sets are the dimensions on which the components of y are greater than (resp. smaller than) x.

A *family* $\mathcal{W}$ *of right-coalition systems* on a compact set $A \in \mathbb{R}^m_+$ satisfies the *intersection property* on A if for any $x \in B(A) \backslash A$, any finite set $\{x^1, \ldots, x^T\} \subseteq A$, any collection $\{W_l \in \mathcal{W}_l(x_l)\}_{l \in \cup_{t=1}^T M^-(x,x^t)}$, and any collection $\{W'_l \in \mathcal{W}_l(x_l)\}_{l \in \cup_{t=1}^T M^+(x,x^t)}$,

$$\cap_{t=1}^T \left\{ \left[\cup_{l \in M^-(x,x^t)} W_l\right] \cup \left[\cup_{l \in M^+(x,x^t)} W'_l\right] \right\} \neq \varnothing.$$

The intersection property was introduced by Barberà, Massó, and Neme (1997) for a finite set of alternatives and extended to the non-finite case by Barberà, Massó, and Serizawa (1998).

A social choice function $F\colon (\mathcal{D}^{CS})^n \to X$ is a *consistent generalized median voting scheme on* $(\mathcal{D}^{CS^*})^n$ if there exists a generalized median voting scheme $\hat{F}\colon (\mathcal{D}^{CS^*})^n \to X$ satisfying the intersection property on A^F for which $\hat{F}(\mathbf{R}) = F(\mathbf{R})$ for all $\mathbf{R} \in (\mathcal{D}^{SS^*})^n$.[8]

[8] By restricting attention to preferences that have a unique top on the range, it is not necessary to introduce strategy-proof tie-breaking rules to choose among the alternatives in anyone's top set.

Theorem 3 shows that the "if" part of Theorem 3.1 in Barberà, Massó, and Serizawa (1998) holds without their assumptions that the tops of the preferences are finite and the range of the social choice function is compact. Theorem 1 is used to help prove Theorem 3.

Theorem 3. *For $m \geq 2$, a strategy-proof social choice function $F\colon (\mathcal{D}^{CS})^n \rightarrow X$ whose range A^F is full-dimensioned is a consistent generalized median voting scheme on $(\mathcal{D}^{CS^*})^n$.*

Proof. Let $\mathcal{D}^{CS'}$ be the restriction of $\mathcal{D}^{CS}$ to the preferences that have tops in X and let F' be the restriction of F to $(\mathcal{D}^{CS'})^n$.

We first show that $A^{F'} = A^F$. Clearly, $A^{F'} \subseteq A^F$. Contrary to what is to be shown, suppose that there exists an $x \in A^F \setminus A^{F'}$. By assumption, $x \in X$. Let R^x be any preference in $\mathcal{D}^{CS'}$ for which $\tau(R^x) = \{x\}$. By Lemma 2, $F(R^x, R^x, \ldots, R^x) = x$. Because $R^x \in \mathcal{D}^{CS'}$, it follows that $F'(R^x, R^x, \ldots, R^x) = x$, contradicting the assumption that $x \notin A^{F'}$. Hence, $A^{F'} = A^F$.

By Theorem 1, the range of F is compact. The function F' therefore has a full-dimensioned compact range. It is also strategy-proof because F is strategy-proof. The restrictions of F and F' to $(\mathcal{D}^{CS^*})^n$ are identical. On this subdomain, by Theorem 3.1 in Barberà, Massó, and Serizawa (1998), the social choice function is a generalized median voting scheme that satisfies the intersection property on $A^{F'}$. Because $A^{F'} = A^F$, the social choice function is also a generalized median voting scheme that satisfies the intersection property on A^F. □

5. Concluding Remarks

We have shown that if the domain conditions of Theorems 1 or 2 or of Corollary 1 are satisfied, then the range of a strategy-proof social choice function must be bounded. To establish these results, we have made essential use of the assumptions that the domain includes preferences whose tops are infinite in one or more dimensions and that a finite amount must be chosen of each public good. These assumptions preclude always choosing a Pareto optimal alternative regardless of whether the social choice function is strategy-proof or not. For example, if the domain includes a preference that is monotonically increasing in some dimension, then any alternative with a finite amount of this good is dominated by one with a larger amount of this good if everybody shares this preference. Consequently, if the top of a preference is permitted to be infinite in any dimension, the standard Pareto optimality condition needs to be relaxed. A natural weakening is to only require the chosen alternative to be Pareto optimal on the range of the social choice function.

Because the domain assumptions that we use to establish that the option sets are compact are not very demanding, it is hoped that our results will facilitate the characterization of social choice functions on many domains of interest by identifying additional structural properties that the option sets must satisfy when strategy-proofness is combined with other desirable properties.

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